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Oscillation constants for second-order nonlinear differential equations with p-Laplacian

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1 Introduction

Consider the nonlinear differential equation

$$(\phi_p(x'))' + \frac{1}{t^p} f(x) = 0, \quad t > 0, \qquad ' = \frac{d}{dt}, \tag{1.1}$$

where $\phi_p(x)$ is the real-valued function defined by $\phi_p(x) = |x|^{p-2}x$ with $p > 1$, and $f(x)$ is a continuous function on $\mathbb{R}$ satisfying

$$xf(x) > 0 \quad \text{if} \quad x \neq 0, \tag{1.2}$$

and a suitable smoothness condition to ensure the uniqueness of solutions of equation (1.1) to the initial value problem. Then each solution of equation (1.1) and its derivative exist in the future, for the proof, see [21, Theorem C]. Hence we can discuss the asymptotic behavior of all solutions of equation (1.1) as $t \to \infty$.

In this paper, we focus on oscillatory behavior of solutions of equation (1.1) as $t \to \infty$. Here a nontrivial solution of equation (1.1) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is said to be *nonoscillatory*.

The research for the oscillatory behavior of equation (1.1) was started by Sugie and Hara [15] two decades ago. They considered equation (1.1) with $p = 2$ and gave a pair of oscillation and nonoscillation theorems. After that, their results were improved by many authors (we refer to [1, 2, 14, 16, 17, 18, 19, 20, 21, 23, 24]). As for the general case $p > 1$, the following oscillation criteria for equation (1.1) were given by Sugie *et al.* [18, 21].

Theorem A ([21, Theorem 1.1]). *Assume* (1.2) *and suppose that there exists* λ *with* $\lambda > \mu_p$ *such that*

$$\frac{f(x)}{\phi_p(x)} \geq \gamma_p + \frac{\lambda}{\log^2(|x|^{p/(p-1)})}$$

for $|x|$ *sufficiently large, where*

$$\gamma_p = \left(\frac{p-1}{p}\right)^p \quad \text{and} \quad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

Then all nontrivial solutions of equation (1.1) *are oscillatory.*

Theorem B ([18, Theorem 1.1]). *Assume* (1.2) *and suppose that*

$$\frac{f(x)}{\phi_p(x)} \leq \gamma_p + \frac{\mu_p}{\log^2(|x|^{p/(p-1)})}$$

for $x > 0$ *or* $x < 0$, *and* $|x|$ *sufficiently large. Then all nontrivial solutions of equation* (1.1) *are nonoscillatory.*

To prove these results, they used the fact that the constant μ_p is the critical value for the oscillation of the Riemann-Weber version of the half-linear differential equation

$$(\phi_p(x'))' + \frac{1}{t^p}\left(\gamma_p + \frac{\lambda}{\log^2 t}\right)\phi_p(x) = 0, \tag{1.3}$$

that is, all nontrivial solutions of equation (1.3) are oscillatory if and only if $\lambda > \mu_p$. Such a number is generally called the *oscillation constant.* We note that there are numerous papers concerning the oscillation constant μ_p for equation (1.3) (e.g., we can refer to [3, 4, 5, 6, 7, 8, 9, 10, 11]).

Let us consider the case that $p = 2$. Then equation (1.3) with $p = 2$ is the Riemann-Weber version of the Euler differential equation. It is known that equation (1.3) with $p = 2$ is equivalent to the linear differential equation

$$x'' + \frac{1}{t^2}\left\{\frac{1}{4} + \sum_{k=1}^{n-1}\frac{1}{4\,\mathrm{Log}_k^2(t)} + \frac{\lambda}{\mathrm{Log}_n^2(t)}\right\}x = 0, \tag{1.4}$$

where

$$\mathrm{Log}_k(t) = \prod_{j=1}^{k}\log_j(t), \qquad \log_k(t) = \log(\log_{k-1}(t)), \quad \log_1(t) = \log t$$

for t sufficiently large, see [12, p. 325], [13] and [22, Theorem 2.42]. Hence all nontrivial solutions of equation (1.4) are oscillatory if and only if $\lambda > \mu_2 = 1/4$.

Remark 1.1. The number $1/4$ is the oscillation constant for equation (1.4).

The oscillation constant for equation (1.4) also plays an essential role in deciding whether or not all nontrivial solutions of equation (1.1) with $p = 2$ are oscillatory or not. In fact, using the oscillation constant for equation (1.4), Sugie and Yamaoka gave the following results.

Theorem C ([20, Lemma 2.3]). *Assume* (1.2) *and suppose that there exist* λ *with* $\lambda > 1/4$ *and* $n \in \mathbb{N}$ *such that*

$$\frac{f(x)}{x} \geq \frac{1}{4} + \sum_{k=1}^{n-1}\frac{1}{4\,\mathrm{Log}_k^2(x^2)} + \frac{\lambda}{\mathrm{Log}_n^2(x^2)} \tag{1.5}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of equation* (1.1) *with* $p = 2$ *are oscillatory.*

Theorem D ([19, Theorem 1.1]). *Assume* (1.2) *and suppose that there exists* $n \in \mathbb{N}$ *such that*

$$\frac{f(x)}{x} \leq \frac{1}{4} + \sum_{k=1}^{n} \frac{1}{4\,\mathrm{Log}_k^2(x^2)} \tag{1.6}$$

for $x > 0$ *or* $x < 0$, *and* $|x|$ *sufficiently large. Then all nontrivial solutions of equation* (1.1) *with* $p = 2$ *are nonoscillatory.*

Here a natural question now arises: what is a pair of oscillation and nonoscillation theorems which extend Theorems A-D? The purpose of this paper is to answer the question. Our results are stated as follows.

Theorem 1.1. *Assume* (1.2) *and suppose that there exist* λ *with* $\lambda > \mu_p$ *and* $n \in \mathbb{N}$ *such that*

$$\frac{f(x)}{\phi_p(x)} \geq \gamma_p + \sum_{k=1}^{n-1} \frac{\mu_p}{\mathrm{Log}_k^2(|x|^{p/(p-1)})} + \frac{\lambda}{\mathrm{Log}_n^2(|x|^{p/(p-1)})} \tag{1.7}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of equation* (1.1) *are oscillatory.*

Theorem 1.2. *Assume* (1.2) *and suppose that there exists* $n \in \mathbb{N}$ *such that*

$$\frac{f(x)}{\phi_p(x)} \leq \gamma_p + \sum_{k=1}^{n} \frac{\mu_p}{\mathrm{Log}_k^2(|x|^{p/(p-1)})} \tag{1.8}$$

for $x > 0$ *or* $x < 0$, *and* $|x|$ *sufficiently large. Then all nontrivial solutions of equation* (1.1) *are nonoscillatory.*

Remark 1.2. When $n = 1$ (resp., $p = 2$), Theorems 1.1 and 1.2 become Theorems A and B (resp., Theorems C and D).

2 Preliminaries

In this section, we prepare some lemmas which is useful for proving our main theorems. To this end, we consider the half-linear differential equation

$$(\phi_p(x'))' + \frac{1}{t^p}\{\gamma_p + \delta(t)\}\,\phi_p(x) = 0 \tag{2.1}$$

and the Riccati inequality

$$\dot{\xi} + (p-1)\,H(\xi, \Gamma_p) + \delta(e^s) \leq 0, \qquad \Gamma_p = 2\mu_p = \left(\frac{p-1}{p}\right)^{p-1}, \qquad \dot{} = \frac{d}{ds}, \tag{2.2}$$

where $\delta(t)$ is a positive continuous function and $H(\xi, G)$ is defined by

$$H(\xi, G) = |\xi + G|^q - q\phi_q(G)\xi - |G|^q, \qquad q = \frac{p}{p-1}.$$

Remark 2.1. For any $\xi, G \in \mathbb{R}$, the function $H(\xi, G)$ is nonnegative. In fact, we see that $H(0, G) = 0$ and

$$\frac{\partial}{\partial \xi} H(\xi, G) = q\phi_q(\xi + G) - q\phi_q(G),$$

which is zero if and only if $\xi = 0$. Then we have $H(\xi, G) > 0$ for $\xi \neq 0$. We also see that, for each fixed G, $H(\xi, G)$ is increasing (resp., decreasing) if $\xi > 0$ (resp., $\xi < 0$). Moreover, from the Taylor expansion of the function $H(\xi, G)$, we see that, for each fixed $G \neq 0$,

$$H(\xi, G) = \frac{q(q-1)|G|^{q-2}}{2}\xi^2 + O(\xi^3)$$

as $\xi \to 0$. Here we use the standard Landau "O" symbol which is defined as follows: $g(t) = O(h(t))$ as $t \to t_0$ if $\limsup_{t\to t_0} |g(t)/h(t)| < \infty$.

To begin with, we show that half-linear differential equation (2.1) have a close relation with differential inequalities of the first order.

Lemma 2.1. *Let $s = \log t$. Suppose that differential inequality* (2.2) *has a solution defined in a neighborhood of ∞. Then all nontrivial solutions of equation* (2.1) *are nonoscillatory.*

Proof. Let $\xi(s)$ be a solution of (2.2) on $[s_0, \infty)$ and define

$$c(s) = -\dot{\xi}(s) - (p-1)\, H(\xi(s), \Gamma_p)$$

for $s \geq s_0$, where s_0 is a large number. Then we have

$$c(s) \geq \delta(e^s) \tag{2.3}$$

for $s \geq s_0$. Let

$$u(s) = \exp\left(\int_{s_0}^{s} \phi_q(\xi(\sigma) + \Gamma_p) d\sigma\right).$$

Then we can check that $u(s)$ is a nonoscillatory solution of the equation

$$(\phi_p(\dot{u}))^{\cdot} - (p-1)\phi_p(\dot{u}) + (\gamma_p + c(s))\, \phi_p(u) = 0.$$

Letting $t = e^s$ and $x(t) = u(s)$, we see that $x(t)$ is a nonoscillatory solution of the equation

$$(\phi_p(x'))' + \frac{1}{t^p}\{\gamma_p + c(\log t)\}\, \phi_p(x) = 0$$

for $t \geq e^{s_0}$. It follows from (2.3) and Sturm's comparison theorem for half-linear differential equations that all nontrivial solutions of equation (2.1) are nonoscillatory. □

Lemma 2.2. *Suppose that the differential inequality*

$$\dot{\xi} + (p-1)H(\xi, \Gamma_p) \leq 0 \tag{2.4}$$

has a solution defined in a neighborhood of ∞. Then this solution is nonincreasing and tends to zero as $s \to \infty$.

Proof. Let $\xi(s)$ be a solution of (2.4) for s sufficiently large. Then we see that $\xi(s)$ is nonincreasing for s sufficiently large because of Remark 2.1. Hence $\xi(s)$ tends to either $-\infty$ or a number as $s \to \infty$.

Suppose that $\xi(s) \to -\infty$ as $s \to \infty$. Since $H(\xi, \Gamma_p) \geq |\xi|^q/2$ for $|\xi|$ sufficiently large because $H(\xi, \Gamma_p)/|\xi|^q \to 1$ as $|\xi| \to \infty$, there exists $s_0 > 0$ such that

$$\dot{\xi}(s) \leq -\frac{p-1}{2}(-\xi(s))^q$$

for $s \geq s_0$. Dividing by $(-\xi(s))^q > 0$ and integrating from s_0 to s, we obtain

$$(-\xi(s))^{1-q} \leq -\frac{1}{2}(s - s_0) + (-\xi(s_0))^{1-q}$$

for $s \geq s_0$. Thus there exists $s_1 > s_0$ such that $\xi(s) \to -\infty$ as $s \to s_1$ from the left, which is a contradiction.

Suppose that there exists a number $\xi_0 \neq 0$ such that $\xi(s) \to \xi_0$ as $s \to \infty$. Then we have

$$\dot{\xi}(s) \leq -(p-1)\, H(\xi(s), \Gamma_p) \leq -(p-1)\, H(\xi_0/2, \Gamma_p) < 0$$

for s sufficiently large. This means that $\xi(s) \to -\infty$ as $s \to \infty$, which is also contradiction. The proof is now complete. □

Lemma 2.3. *Suppose that $\xi(s)$ satisfies the differential inequality*

$$\dot{\xi}(s) + (p-1)H(\xi(s), \Gamma_p) + \frac{\lambda}{s^2} \leq 0 \tag{2.5}$$

for s sufficiently large, where λ is a positive constant. Then there exists $M > 0$ such that

$$\xi(s) \leq \frac{2\Gamma_p}{s} + \frac{M}{s^2}$$

for s sufficiently large.

Proof. Let

$$\Omega(s) = \Gamma_p\, s^2 \left(1 + \frac{2}{(p-1)s}\right)^{p-1}, \quad U(s) = -\Gamma_p\, s^2 + \Omega(s), \quad \eta(s) = s^2\xi(s) - U(s).$$

Then we see that

$$\begin{aligned} U(s) =& -\Gamma_p\, s^2 + \Gamma_p\, s^2 \left(1 + \frac{2}{(p-1)s}\right)^{p-1} \\ =& -\Gamma_p\, s^2 + \Gamma_p\, s^2 \left\{1 + \frac{2}{s} + \frac{2(p-2)}{(p-1)s^2} + O\left(\frac{1}{s^3}\right)\right\} \\ =& \Gamma_p \left\{2s + \frac{2(p-2)}{p-1} + O\left(\frac{1}{s}\right)\right\} \end{aligned}$$

as $s \to \infty$. Therefore, by a direct computation, we get

$$\begin{aligned}
\dot{\eta}(s) =& s^2\dot{\xi}(s) + 2s\xi(s) - \dot{U}(s) \\
\le& s^2\left[-(p-1)H(\xi(s),\Gamma_p) - \frac{\lambda}{s^2}\right] + 2s\frac{\eta(s)+U(s)}{s^2} - \dot{U}(s) \\
=& -(p-1)s^2 H\left(\frac{\eta(s)+U(s)}{s^2},\Gamma_p\right) - \lambda + \frac{2}{s}\eta(s) + \left(\frac{2}{s}U(s) - \dot{U}(s)\right) \\
=& -(p-1)s^2\left\{\left|\frac{\eta(s)+U(s)}{s^2}+\Gamma_p\right|^q - \frac{\eta(s)+U(s)}{s^2} - \gamma_p\right\} \\
& - \lambda + \frac{2}{s}\eta(s) + 2\Gamma_p\left(1+\frac{2}{(p-1)s}\right)^{p-2} \\
=& -(p-1)s^{2(1-q)}\left|\eta(s)+\Omega(s)\right|^q + (p-1)\eta(s) + (p-1)U(s) + (p-1)\gamma_p s^2 \\
& - \lambda + \frac{2}{s}\eta(s) + 2\Gamma_p + O\left(\frac{1}{s}\right) \\
=& -(p-1)s^{2(1-q)}\left|\eta(s)+\Omega(s)\right|^q + (p-1)\left(1+\frac{2}{(p-1)s}\right)\eta(s) \\
& + (p-1)\Gamma_p\left\{2s + \frac{2(p-2)}{p-1}\right\} + (p-1)\gamma_p s^2 - \lambda + 2\Gamma_p + O\left(\frac{1}{s}\right) \\
=& -(p-1)s^{2(1-q)}\left\{\left|\eta(s)+\Omega(s)\right|^q\right. \\
& \left. - s^{2(q-1)}\left(1+\frac{2}{(p-1)s}\right)\eta(s) - |\Omega(s)|^q + |\Omega(s)|^q\right\} \\
& + \Gamma_p\left\{2(p-1)s + 2(p-2) + 2\right\} + (p-1)\gamma_p s^2 - \lambda + O\left(\frac{1}{s}\right) \\
=& -(p-1)s^{2(1-q)}H(\eta(s),\Omega(s)) - (p-1)s^{2(1-q)}|\Omega(s)|^q \\
& + 2(p-1)\Gamma_p(s+1) + (p-1)\gamma_p s^2 - \lambda + O\left(\frac{1}{s}\right) \\
=& -(p-1)s^{2(1-q)}H(\eta(s),\Omega(s)) - (p-1)\gamma_p s^2\left(1+\frac{2}{(p-1)s}\right)^p \\
& + 2p\gamma_p(s+1) + (p-1)\gamma_p s^2 - \lambda + O\left(\frac{1}{s}\right) \\
=& -(p-1)s^{2(1-q)}H(\eta(s),\Omega(s)) - (p-1)\gamma_p s^2\left\{1+\frac{2p}{(p-1)s}+\frac{2p}{(p-1)s^2}\right\} \\
& + 2p\gamma_p(s+1) + (p-1)\gamma_p s^2 - \lambda + O\left(\frac{1}{s}\right) \\
=& -(p-1)s^{2(1-q)}H(\eta(s),\Omega(s)) - \lambda + O\left(\frac{1}{s}\right)
\end{aligned}$$

as $s \to \infty$. It follows from Remark 2.1 and positivity of the constant λ that $\dot{\eta}(s) < 0$ for s

sufficiently large, and therefore, there exists s_0 such that $\eta(s) \leq \eta(s_0)$ for $s \geq s_0$. Since

$$\begin{aligned}\xi(s) =&\frac{U(s)+\eta(s)}{s^2} = \Gamma_p\left\{\frac{2}{s}+\frac{2(p-2)}{(p-1)s^2}+O\left(\frac{1}{s^3}\right)\right\}+\frac{\eta(s)}{s^2}\\ =&\frac{2\Gamma_p}{s}+\frac{\eta(s)}{s^2}+O\left(\frac{1}{s^2}\right)\end{aligned}$$

as $s \to \infty$, we can find $M_1 > 0$ and $s_1 \geq s_0$ such that

$$\xi(s) \leq \frac{2\Gamma_p}{s}+\frac{M_1+\eta(s_0)}{s^2}$$

for $s \geq s_1$. □

Remark 2.2. Suppose that $\xi(s)$ satisfies (2.5) for s sufficiently large. Then, from Lemma 2.2, we see that $\xi(s) > 0$ for s sufficiently large. Hence, together with Lemma 2.3, we can show that $\xi(s) = O(1/s)$ as $s \to \infty$.

We next show that the oscillation constant for the half-linear differential equation

$$(\phi(x'))' + \frac{1}{t^p}\left\{\gamma_p + \delta_n(t)\right\}\phi_p(x) = 0 \tag{2.6}$$

is μ_p, where

$$\delta_n(t) = \sum_{k=1}^{n-1}\frac{\mu_p}{\mathrm{Log}_k^2(t)}+\frac{\lambda}{\mathrm{Log}_n^2(t)}.$$

Lemma 2.4. *Let $n \in \mathbb{N}$. Then all nontrivial solutions of equation* (2.6) *are oscillatory if and only if $\lambda > \mu_p$.*

Proof. We first prove 'if' part. Let $\lambda > \mu_p$. Then there exists $\varepsilon_0 > 0$ such that

$$\lambda - \varepsilon_0 > \mu_p. \tag{2.7}$$

By way of contradiction, we suppose that equation (2.6) has a nonoscillatory solution $x(t)$. Let $s = \log t$ and $u(s) = x(t)$. Then equation (2.6) becomes the equation

$$(\phi_p(\dot{u}))^{\cdot} - (p-1)\phi_p(\dot{u}) + \left\{\gamma_p + \delta_n(e^s)\right\}\phi_p(u) = 0. \tag{2.8}$$

Define

$$\xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))} - \Gamma_p.$$

Then $\xi(s)$ satisfies

$$\dot{\xi}(s) =\frac{(\phi_p(\dot{u}(s)))^{\cdot}\phi_p(u(s)) - (p-1)\phi_p(\dot{u}(s))|u(s)|^{p-2}\dot{u}(s)}{\phi_p(u(s))^2}$$

$$
\begin{aligned}
&=\frac{(\phi_p(\dot{u}(s)))^{\cdot}}{\phi_p(u(s))}-(p-1)\left|\frac{\dot{u}(s)}{u(s)}\right|^p\\
&=(p-1)\frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))}-\gamma_p-\delta_n(e^s)-(p-1)\left|\frac{\dot{u}(s)}{u(s)}\right|^{(p-1)q}\\
&=(p-1)(\xi(s)+\Gamma_p)-\gamma_p-\delta_n(e^s)-(p-1)|\xi(s)+\Gamma_p|^q\\
&=-(p-1)\left\{|\xi(s)+\Gamma_p|^q-(\xi(s)+\Gamma_p)+\frac{\gamma_p}{p-1}\right\}-\delta_n(e^s)\\
&=-(p-1)\left\{|\xi(s)+\Gamma_p|^q-\xi(s)-\gamma_p\right\}-\delta_n(e^s)\\
&=-(p-1)\left\{|\xi(s)+\Gamma_p|^q-q\phi_q(\Gamma_p)\xi(s)-|\Gamma_p|^q\right\}-\delta_n(e^s)\\
&=-(p-1)H(\xi(s),\Gamma_p)-\delta_n(e^s)
\end{aligned}
$$

for s sufficiently large, and therefore, from Lemma 2.3 and Remark 2.2, we see that $\xi(s)=O(1/s)$ as $s\to\infty$. Hence, together with the Taylor expansion of the function $H(\xi,\Gamma_p)$ (see Remark 2.1) and the relation $(p-1)(q-1)=1$, we have

$$
\begin{aligned}
\dot{\xi}(s)&=-(p-1)\left\{\frac{q(q-1)|\Gamma_p|^{q-2}}{2}\xi^2(s)+O(\xi^3(s))\right\}-\delta_n(e^s)\\
&=-\frac{q|\Gamma_p|^{q-2}}{2}\xi^2(s)-\delta_n(e^s)+O(\xi^3(s))\\
&=-\frac{q\phi_q(\Gamma_p)}{2\Gamma_p}\xi^2(s)-\left(\sum_{k=1}^{n-1}\frac{\mu_p}{\mathrm{Log}_k^2(e^s)}+\frac{\lambda}{\mathrm{Log}_n^2(e^s)}\right)+O\left(\frac{1}{s^3}\right)\\
&\le-\frac{1}{4\mu_p}\xi^2(s)-\left(\sum_{k=1}^{n-1}\frac{\mu_p}{\mathrm{Log}_k^2(e^s)}+\frac{\lambda-\varepsilon_0}{\mathrm{Log}_n^2(e^s)}\right)
\end{aligned}
$$

for s sufficiently large. Let

$$\xi_1(s)=\frac{1}{4\mu_p}\xi(s).$$

Then $\xi_1(s)$ satisfies

$$\dot{\xi}_1(s)\le-\xi_1^2(s)-\left(\sum_{k=1}^{n-1}\frac{1}{4\,\mathrm{Log}_k^2(e^s)}+\frac{\lambda-\varepsilon_0}{4\mu_p}\frac{1}{\mathrm{Log}_n^2(e^s)}\right)$$

for s sufficiently large. We note that $H(\xi,\Gamma_p)=\xi^2$ and $\gamma_p=1/4$ when $p=2$. Hence, it follows from Lemma 2.1 with $p=2$ that all nontrivial solutions of the linear equation

$$x''+\frac{1}{t^2}\left\{\frac{1}{4}+\sum_{k=1}^{n-1}\frac{1}{4\,\mathrm{Log}_k^2(t)}+\frac{\lambda-\varepsilon_0}{4\mu_p}\frac{1}{\mathrm{Log}_n^2(t)}\right\}x=0$$

are nonoscillatory. On the other hand, from Remark 1.1, we get

$$\frac{\lambda-\varepsilon_0}{4\mu_p}\le\frac{1}{4},$$

which is a contradiction to (2.7). Thus all nontrivial solutions of equation (2.6) are oscillatory if $\lambda > \mu_p$.

We next show 'only-if' part. Using Remark 1.1 again, we see that all nontrivial solutions of the linear equation

$$y'' + \frac{1}{t^2}\left\{\frac{1}{4} + \sum_{k=1}^{n+1}\frac{1}{4\operatorname{Log}_k^2(t)}\right\}y = 0$$

are nonoscillatory. Let $y(t)$ be a nontrivial solution of this equation. Put $s = \log t$ and $v(s) = y(t)$. Then $v(s)$ satisfies

$$\ddot{v} - \dot{v} + \left\{\frac{1}{4} + \sum_{k=1}^{n+1}\frac{1}{4\operatorname{Log}_k^2(e^s)}\right\}v = 0,$$

and therefore, by putting

$$\eta(s) = \frac{\dot{v}(s)}{v(s)} - \frac{1}{2},$$

we see that $\eta(s)$ satisfies

$$\dot{\eta}(s) = -\eta^2(s) - \sum_{k=1}^{n+1}\frac{1}{4\operatorname{Log}_k^2(e^s)}$$

for s sufficiently large. Hence, using Lemma 2.3 with $p = 2$ and Remark 2.2, we get $\eta(s) = O(1/s)$ as $s \to \infty$. Let $\eta_1(s) = 4\mu_p\eta(s)$. Then, together with Remark 2.1, we see that $\eta_1(s)$ satisfies

$$\begin{aligned}
\dot{\eta}_1(s) &= -\frac{1}{4\mu_p}\eta_1^2(s) - \sum_{k=1}^{n+1}\frac{\mu_p}{\operatorname{Log}_k^2(e^s)} \\
&= -(p-1)\frac{q(q-1)|\Gamma_p|^{q-2}}{2}\eta_1^2(s) - \sum_{k=1}^{n+1}\frac{\mu_p}{\operatorname{Log}_k^2(e^s)} \\
&= -(p-1)H(\eta_1(s),\Gamma_p) - \sum_{k=1}^{n+1}\frac{\mu_p}{\operatorname{Log}_k^2(e^s)} + O\left(\eta_1^3(s)\right) \\
&= -(p-1)H(\eta_1(s),\Gamma_p) - \sum_{k=1}^{n+1}\frac{\mu_p}{\operatorname{Log}_k^2(e^s)} + O\left(\frac{1}{s^3}\right)
\end{aligned}$$

as $s \to \infty$. Hence we have

$$\begin{aligned}
\dot{\eta}_1(s) &\le -(p-1)H(\eta_1(s),\Gamma_p) - \sum_{k=1}^{n}\frac{\mu_p}{\operatorname{Log}_k^2(e^s)} \\
&\le -(p-1)H(\eta_1(s),\Gamma_p) - \left(\sum_{k=1}^{n-1}\frac{\mu_p}{\operatorname{Log}_k^2(e^s)} + \frac{\lambda}{\operatorname{Log}_n^2(e^s)}\right)
\end{aligned}$$

$$= -\,(p-1)H(\eta_1(s),\Gamma_p) - \delta_n(e^s)$$

for s sufficiently large if $\lambda \le \mu_p$. Thus, from Lemma 2.1, we see that all nontrivial solutions of equation (2.6) are nonoscillatory when $\lambda \le \mu_p$. This completes the proof. □

Remark 2.3. If $\xi(s)$ satisfies the differential inequality

$$\dot{\xi}(s) \le -(p-1)\,H(\xi(s),\Gamma_p) - \left\{\sum_{k=1}^{n-1} \frac{\mu_p}{\mathrm{Log}_k^2(e^s)} + \frac{\lambda}{\mathrm{Log}_n^2(e^s)}\right\}$$

for s sufficiently large, then from Lemma 2.1 all nontrivial solutions of equation (2.6) are nonoscillatory. Hence, in view of Lemma 2.4, we have $\lambda \le \mu_p$.

In the next lemma, we estimate the asymptotic behavior of nonoscillatory solutions of equation (2.6). This asymptotic behavior will be useful to prove Theorem 1.2.

Lemma 2.5. *Suppose that equation* (2.6) *has a nonoscillatory solution. Then there exists the solution $y_H(t)$ of equation* (2.6) *such that*

$$y_H(t) \ge t^{(p-1)/p} \quad \text{and} \quad t y_H'(t) > \frac{p-1}{p} y_H(t) \tag{2.9}$$

for t sufficiently large.

Proof. Let $y(t)$ be a nonoscillatory solution of equation (2.6). Then, without loss of generality, we may assume that $y(t)$ is positive for t sufficiently large. Put $s = \log t$, $u(s) = y(t)$ and $\xi(s) = \phi_p(\dot{u}(s))/\phi_p(u(s)) - \Gamma_p$. Then we have

$$\dot{\xi}(s) + (p-1)H(\xi(s),\Gamma_p) + \delta_n(e^s) = 0. \tag{2.10}$$

Hence, from Lemma 2.2 and positivity of the function $\delta_n(e^s)$, we see that $\xi(s)$ is decreasing and tends to zero as $s \to \infty$, and therefore, we have

$$\frac{\dot{u}(s)}{u(s)} > \frac{p-1}{p} \tag{2.11}$$

for s sufficiently large. Hence there exists a positive constant M such that

$$\log u(s) \ge \frac{p-1}{p} s - M,$$

and therefore, we obtain

$$y(t) = u(s) \ge e^{-M} e^{(p-1)s/p} = e^{-M} t^{(p-1)/p}$$

for t sufficiently large. Here we put $y_H(t) = y(t)/e^{-M}$. Since equation (2.6) is a half-linear differential equation, $y_H(t)$ is also a solution of equation (2.6) satisfying $y_H(t) \ge t^{(p-1)/p}$ for t sufficiently large. We also see that $t y_H'(t) > (p-1)y_H(t)/p$ for t sufficiently large because of (2.11). □

3 Proof of the main theorems

In this section, we give the proofs of oscillation criteria for equation (1.1). Using the following lemma, we first prove the oscillation theorem, Theorem 1.1.

Lemma 3.1 ([21, Lemma 3.1]). *Assume* (1.2) *and suppose that equation* (1.1) *has a positive solution. Then it is increasing for t sufficiently large and it tends to ∞ as $t \to \infty$.*

Proof of Theorem 1.1. The proof is by contradiction. Suppose that equation (1.1) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is positive for t sufficiently large. Let L be so large number that (1.7) is satisfied for $|x| > L$. By Lemma 3.1, we have $x(t) > L$ and $x'(t) > 0$ for t sufficiently large.

Let $s = \log t$ and $u(s) = x(t)$. Then equation (1.1) is transformed into the equation

$$(\phi_p(\dot{u}))^{\cdot} - (p-1)\phi_p(\dot{u}) + f(u) = 0.$$

Moreover, we see that $u(s) > L$ and $\dot{u}(s) = tx'(t) > 0$ for s sufficiently large. Define

$$\xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))} - \Gamma_p. \tag{3.1}$$

Differentiating $\xi(s)$ and using (1.7), we have

$$\begin{aligned}
\dot{\xi}(s) &= (p-1)\frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))} - \frac{f(u(s))}{\phi_p(u(s))} - (p-1)\left|\frac{\dot{u}(s)}{u(s)}\right|^{(p-1)q} \\
&\le (p-1)(\xi(s)+\Gamma_p) - \gamma_p - \delta_n\left(u(s)^{p/(p-1)}\right) - (p-1)|\xi(s)+\Gamma_p|^q \\
&= -(p-1)\,H(\xi(s),\Gamma_p) - \delta_n\left(u(s)^{p/(p-1)}\right)
\end{aligned} \tag{3.2}$$

for s sufficiently large, where $\delta_n(t)$ is the same function as in equation (2.6). Hence, from Lemma 2.2, we see that $\xi(s) \searrow 0$ as $s \to \infty$, and therefore, using (3.1), we have

$$\frac{\dot{u}(s)}{u(s)} \searrow \frac{p-1}{p} \quad \text{as } s \to \infty. \tag{3.3}$$

Since $\lambda > \mu_p$, we can choose $\varepsilon_0 > 0$ so small that

$$\lambda - \varepsilon_0 > \mu_p. \tag{3.4}$$

By (3.3), we see that

$$\frac{\dot{u}(s)}{u(s)} \le \frac{p-1}{p}\left(1+\frac{\varepsilon_0}{2}\right)$$

for s sufficiently large, and therefore, we obtain

$$\log u(s) \le \frac{p-1}{p}(1+\varepsilon_0)s$$

for s sufficiently large. From this inequality and (3.2), we get

$$\begin{aligned}\dot{\xi}(s) &\leq -(p-1)\,H(\xi(s),\Gamma_p) - \delta_n\left(u(s)^{p/(p-1)}\right)\\ &\leq -(p-1)\,H(\xi(s),\Gamma_p) - \frac{\mu_p}{\log^2(u(s)^{p/(p-1)})}\\ &\leq -(p-1)\,H(\xi(s),\Gamma_p) - \frac{\mu_p}{(1+\varepsilon_0)^2\,s^2}\end{aligned}$$

for s sufficiently large. It follows from Lemma 2.3 that there exists $M > 0$ such that

$$\phi_p\left(\frac{\dot{u}(s)}{u(s)}\right) = \Gamma_p + \xi(s) \leq \Gamma_p + \frac{2\Gamma_p}{s} + \frac{M\Gamma_p}{s^2} = \Gamma_p\left(1 + \frac{2}{s} + \frac{M}{s^2}\right),$$

and therefore, we can find $M_1 > 0$ such that

$$\frac{\dot{u}(s)}{u(s)} \leq \frac{p-1}{p}\left(1 + \frac{2}{s} + \frac{M}{s^2}\right)^{1/(p-1)} \leq \frac{p-1}{p} + \frac{2}{ps} + \frac{M_1}{s^2}$$

for s sufficiently large. Thus there exists $M_2 > 0$ such that

$$\log u(s) \leq \frac{p-1}{p}(s + M_2 \log s)$$

for s sufficiently large. Hence we get

$$\log_j\left(u(s)^{p/(p-1)}\right) \leq (\log_j(e^s))\left(1 + \frac{M_2 \log s}{s}\right) \tag{3.5}$$

for $j = 1, 2, \ldots, n$. In fact, we can easily check (3.5) by using mathematical induction on j. It is clear that (3.5) is true for $j = 1$. Assume that (3.5) with $j = i$ holds. Then

$$\begin{aligned}\log_{i+1}\left(u(s)^{p/(p-1)}\right) &= \log\left(\log_i\left(u(s)^{p/(p-1)}\right)\right)\\ &\leq \log\left((\log_i(e^s))\left(1 + \frac{M_2\log s}{s}\right)\right)\\ &= \log_{i+1}(e^s) + \log\left(1 + \frac{M_2 \log s}{s}\right)\\ &\leq \log_{i+1}(e^s) + \frac{M_2 \log s}{s}\\ &\leq (\log_{i+1}(e^s))\left(1 + \frac{M_2 \log s}{s}\right)\end{aligned}$$

for s sufficiently large. Thus, (3.5) with $j = i + 1$ is true. Hence we get the equality

$$\begin{aligned}\mathrm{Log}_k\left(u(s)^{p/(p-1)}\right) &= \prod_{j=1}^{k} \log_j\left(u(s)^{p/(p-1)}\right)\\ &\leq \prod_{j=1}^{k}\left\{(\log_j(e^s))\left(1 + \frac{M_2 \log s}{s}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&= (\mathrm{Log}_k(e^s)) \left(1 + \frac{M_2 \log s}{s}\right)^k \\
&= (\mathrm{Log}_k(e^s)) \left\{1 + \frac{kM_2 \log s}{s} + O\left(\left(\frac{\log s}{s}\right)^2\right)\right\} \\
&\le (\mathrm{Log}_k(e^s)) \left(1 + \frac{(n+1)M_2 \log s}{s}\right)
\end{aligned}
$$

for $k = 1, 2, \ldots, n$. Using (3.2), we have

$$
\begin{aligned}
\dot{\xi}(s) \le &- (p-1)\, H(\xi(s), \Gamma_p) - \left\{\sum_{k=1}^{n-1} \frac{\mu_p}{\mathrm{Log}_k^2(u(s)^{p/(p-1)})} + \frac{\lambda}{\mathrm{Log}_n^2(u(s)^{p/(p-1)})}\right\} \\
\le &- (p-1)\, H(\xi(s), \Gamma_p) - \left\{\sum_{k=1}^{n-1} \frac{\mu_p}{\mathrm{Log}_k^2(e^s)} + \frac{\lambda}{\mathrm{Log}_n^2(e^s)}\right\} \left(1 + \frac{(n+1)M_2 \log s}{s}\right)^{-2} \\
= &- (p-1)\, H(\xi(s), \Gamma_p) - \left\{\sum_{k=1}^{n-1} \frac{\mu_p}{\mathrm{Log}_k^2(e^s)} + \frac{\lambda}{\mathrm{Log}_n^2(e^s)}\right\} \left(1 + O\left(\frac{\log s}{s}\right)\right) \\
= &- (p-1)\, H(\xi(s), \Gamma_p) - \left\{\sum_{k=1}^{n-1} \frac{\mu_p}{\mathrm{Log}_k^2(e^s)} + \frac{\lambda}{\mathrm{Log}_n^2(e^s)}\right\} + O\left(\frac{\log s}{s^3}\right)
\end{aligned}
$$

as $s \to \infty$. Hence we get

$$
\dot{\xi}(s) \le -(p-1)\, H(\xi(s), \Gamma_p) - \left\{\sum_{k=1}^{n-1} \frac{\mu_p}{\mathrm{Log}_k^2(e^s)} + \frac{\lambda - \varepsilon_0}{\mathrm{Log}_n^2(e^s)}\right\}
$$

for s sufficiently large. By Remark 2.3, we have $\lambda - \varepsilon_0 \le \mu_p$, which is a contradiction to (3.4). The proof of Theorem 1.1 is now complete. □

We next prove the nonoscillation theorem, Theorem 1.2. To this end, we prepare some useful lemmas. Let $s = \log t$ and $u(s) = x(t)$. Then equation (1.1) is equivalent to the system

$$
\dot{u} = \phi_q(v), \quad \dot{v} = (p-1)v - f(u), \quad q = \frac{p}{p-1}. \tag{3.6}
$$

Here we call the projection of a positive semitrajectory of system (3.6) onto the phase plane a *positive orbit*. For convenience, we write the positive orbit of system (3.6) starting at a point $P \in \mathbb{R}^2$ as $\Gamma_{(3.6)}(P)$.

Lemma 3.2. *Assume* (1.2) *and suppose that equation* (1.1) *has a nontrivial oscillatory solution. Then the positive orbit of system* (3.6) *corresponding to this solution rotates around the origin in the clockwise direction as s increases.*

Proof. Let $x(t)$ be a nontrivial oscillatory solution of equation (1.1). Then $x(t)$ has the infinite number of zeros $\{t_n\}$ clustering at $t = \infty$. Let $(u(s), v(s))$ be the solution of system (3.6) which corresponds to $x(t)$. Then we see that

$$
(u(s), v(s)) = (x(e^s), \phi_p(e^s x'(e^s))).
$$

Hence we have

$$u(s_n) = 0 \tag{3.7}$$

for $n \in \mathbb{N}$, where $s_n = \log t_n$. We also have $\dot{u}(s_n) \neq 0$ for $n \in \mathbb{N}$. In fact, if there exists $m \in \mathbb{N}$ such that $\dot{u}(s_m) = 0$, then we obtain $v(s_m) = \phi_p(\dot{u}(s_m)) = 0$. Since the origin is the unique equilibrium of system (3.6), we have $(u(s), v(s)) = (0, 0)$ for $s \geq s_m$. This contradicts the fact that $x(t)$ is a nontrivial solution of equation (1.1). Thus $u(s)$ changes its sign at $s = s_n$.

We may assume without loss of generality that

$$u(s) < 0 \quad \text{if} \ \ s_{2k-1} < s < s_{2k}, \tag{3.8}$$

$$u(s) > 0 \quad \text{if} \ \ s_{2k} < s < s_{2k+1}, \tag{3.9}$$

$$\dot{u}(s_{2k-1}) < 0 \quad \text{and} \quad \dot{u}(s_{2k}) > 0 \tag{3.10}$$

for $k \in \mathbb{N}$. By (3.10), we have

$$v(s_{2k-1}) = \phi_p(\dot{u}(s_{2k-1})) < 0 \quad \text{and} \quad v(s_{2k}) = \phi_p(\dot{u}(s_{2k})) > 0. \tag{3.11}$$

From the continuity of $v(s)$, we see that $v(s)$ has at least one zero in the interval (s_{2k-1}, s_{2k}) for each $k \in \mathbb{N}$. Let τ be a zero of $v(s)$ belonging to (s_{2k-1}, s_{2k}). Then it follows from (3.8) that $u(\tau) < 0$, and therefore, by (1.2), we have $\dot{v}(\tau) = (p-1)v(\tau) - f(u(\tau)) > 0$, which means that $v(s)$ has only one zero between s_{2k-1} and s_{2k} because of (3.11). Similarly $v(s)$ also has only one zero between s_{2k} and s_{2k+1}. Thus, for any $k \in \mathbb{N}$, there exist $\tilde{s}_{2k-1}$ and $\tilde{s}_{2k}$ with $s_{2k-1} < \tilde{s}_{2k-1} < s_{2k} < \tilde{s}_{2k} < s_{2k+1}$ such that

$$v(\tilde{s}_{2k-1}) = v(\tilde{s}_{2k}) = 0. \tag{3.12}$$

Consider the positive orbit of system (3.6) corresponding to $(u(s), v(s))$. Then, from (3.7)-(3.12), we see that the positive orbit crosses axes in the following order: the negative v-axis at $s = s_{2k-1}$; the negative u-axis at $s = \tilde{s}_{2k-1}$; the positive v-axis at $s = s_{2k}$; the positive u-axis at $s = \tilde{s}_{2k}$. In other words, the positive orbit rotates around the origin in the clockwise direction as s increases. □

Lemma 3.3. *Assume* (1.2) *and suppose that equation* (1.1) *has a nontrivial oscillatory solution* $x(t)$. *Let* $(u(s), v(s))$ *be the solution of system* (3.6) *corresponding to* $x(t)$. *Then* $(u(s), v(s))$ *is unbounded.*

Proof. The proof is by contradiction. Suppose that $(u(s), v(s))$ is bounded, that is, there exist $K > 0$ and $s_0 > 0$ such that $u^2(s) + v^2(s) < K^2$ for $s \geq s_0$.

Define the Lyapunov function

$$V(u, v) = \frac{|v|^q}{q} + \int_0^u f(\chi) d\chi.$$

Then we have

$$\begin{aligned}\frac{d}{ds}V(u(s),v(s)) =&\phi_q(v(s))\{(p-1)v(s) - f(u(s))\} + f(u(s))\phi_q(v(s))\\ =&(p-1)|v(s)|^q.\end{aligned}$$

Since $V(u(s),v(s))$ is nondecreasing for $s \geq s_0$, we have

$$V(u(s),v(s)) \geq V(u(s_0),v(s_0)) =: V_0$$

for $s \geq s_0$. On the other hand, there exists $V_\infty > 0$ such that $V(u(s),v(s)) \to V_\infty$ as $s \to \infty$ because $(u(s),v(s))$ is bounded. Hence we have

$$0 < V_0 \leq V(u(s),v(s)) \leq V_\infty < \infty, \tag{3.13}$$

that is,

$$(u(s),v(s)) \notin \{(u,v) \in \mathbb{R}^2 \,:\, V(u,v) < V_0\} =: R_0$$

for $s \geq s_0$. Note that R_0 is the region which contains an open ball centered at the origin. Hence we can find ε_0 so small that

$$\{(u,v) \,:\, |u| < \varepsilon_0 \text{ and } |v| < \varepsilon_0\} \subset R_0.$$

Since the positive orbit of system (3.6) corresponding to $(u(s),v(s))$ rotates around the region R_0 in the clockwise direction as s increases, there exist sequences $\{\sigma_n\}$ and $\{\tau_n\}$ with $s_0 < \sigma_n < \tau_n < \sigma_{n+1}$ and $\sigma_n \to \infty$ as $n \to \infty$ such that

$$\begin{aligned}&u(\sigma_n) = 0, \quad v(\sigma_n) > \varepsilon_0, \quad u(\tau_n) > \varepsilon_0, \quad v(\tau_n) = \varepsilon_0 \text{ and}\\ &\varepsilon_0 < v(s) < (qV_\infty)^{1/q} =: K \text{ for } \sigma_n < s < \tau_n.\end{aligned}$$

Hence we have

$$\varepsilon_0 < u(\tau_n) - u(\sigma_n) = \int_{\sigma_n}^{\tau_n} \dot{u}(s)ds = \int_{\sigma_n}^{\tau_n} \phi_q(v(s))ds < \phi_q(K)(\tau_n - \sigma_n),$$

and therefore, we obtain

$$\begin{aligned}V(u(s),v(s)) - V_0 = V(u(s),v(s)) - V(u(s_0),v(s_0)) &= \int_{s_0}^{s} \frac{d}{d\sigma}V(u(\sigma),v(\sigma))d\sigma\\ \geq (p-1)\sum_{k=1}^{n}\int_{\sigma_n}^{\tau_n} |v(s)|^q ds &> (p-1)\varepsilon_0^q \sum_{k=1}^{n}(\tau_n - \sigma_n)\\ > \frac{(p-1)\varepsilon_0^{q+1}}{\phi_q(K)}n\end{aligned}$$

for $s \geq \tau_n$. From (3.13), we have

$$V_\infty - V_0 > \frac{(p-1)\varepsilon_0^{q+1}}{\phi_q(K)}n \to \infty$$

as $n \to \infty$, which is a contradiction. Thus, the lemma is proved. □

From Lemmas 3.2 and 3.3, we have the following lemma.

Lemma 3.4. *Assume* (1.2) *and suppose that equation* (1.1) *has a nontrivial oscillatory solution. Then all nontrivial positive orbits of system* (3.6) *rotate around the origin in the clockwise direction as s increases.*

Proof. Let $x(t)$ be a nontrivial oscillatory solution of equation (1.1). Then, it follows from Lemmas 3.2 and 3.3 that the positive orbit of system (3.6) corresponding to $x(t)$ rotates around the origin in the clockwise direction, and runs to infinity as $s \to \infty$. Since system (3.6) is autonomous, the positive orbit is not intersected by any other positive orbits of system (3.6). Hence all nontrivial positive orbits of system (3.6) rotate around the origin in the clockwise direction as s increases. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We give only the proof of the case that (1.8) holds for $x > L$, where L is a large number. Because the other case is carried out in the same manner.

To begin with, we consider half-linear differential equation (2.6) with $\lambda = \mu_p$. Then, from Lemmas 2.4 and 2.5, there exists the solution $y_H(t)$ of equation (2.6) with $\lambda = \mu_p$ such that $y_H(t) \ge t^{(p-1)/p}$ and $ty_H'(t) > (p-1)y_H(t)/p$ for t sufficiently large. Put $s = \log t$ and $(u_H(s), v_H(s)) = (y_H(t), \phi_p(ty_H'(t)))$. Then $(u_H(s), v_H(s))$ satisfies the system

$$\dot{u} = \phi_q(v), \quad \dot{v} = (p-1)v - \left\{\gamma_p + \sum_{k=1}^{n} \frac{\mu_p}{\mathrm{Log}_k^2(e^s)}\right\} \phi_p(u)$$

for s sufficiently large. We also see that there exists $s_0 > 0$ such that

$$u_H(s) \ge e^{(p-1)s/p} > L \quad \text{and} \quad v_H(s) > \Gamma_p \phi_p(u_H(s)) \tag{3.14}$$

for $s \ge s_0$. Now we put $\xi_H(s) = v_H(s)/\phi_p(u_H(s)) - \Gamma_p$. Then $\xi_H(s)$ satisfies

$$\dot{\xi} = -(p-1)\, H(\xi, \Gamma_p) - \sum_{k=1}^{n} \frac{\mu_p}{\mathrm{Log}_k^2(e^s)} \tag{3.15}$$

and $\xi_H(s) > 0$ for $s \ge s_0$.

Suppose that equation (1.1) has a nontrivial oscillatory solution. Then, from Lemma 3.4, all nontrivial positive orbits of system (3.6) rotate around the origin in the clockwise direction as s increases. Let $(u(s), v(s))$ be a nontrivial solution of system (3.6) satisfying

$$(u(s_0), v(s_0)) = (u_H(s_0), v_H(s_0)) \in \{(u,v) \mid u > L, v > \Gamma_p \phi_p(u)\}. \tag{3.16}$$

Then the positive orbit corresponding to $(u(s), v(s))$ also rotates around the origin in the clockwise direction as s increases, and therefore, there exists $s_1 > s_0$ such that

$$\frac{v(s)}{\phi_p(u(s))} > \Gamma_p \quad \text{for} \quad s_0 \le s < s_1 \quad \text{and} \quad \frac{v(s)}{\phi_p(u(s))} = \Gamma_p \quad \text{at} \quad s = s_1. \tag{3.17}$$

Then we have $\dot{u}(s)/u(s) \geq (p-1)/p$ for $s_0 \leq s \leq s_1$. Hence, together with (3.14) and (3.16), we have

$$\begin{aligned}\log u(s) &\geq \frac{p-1}{p}(s-s_0) + \log u(s_0) = \frac{(p-1)}{p}s + \log \frac{u_H(s_0)}{e^{(p-1)s_0/p}} \\ &\geq \frac{(p-1)}{p}s\end{aligned}$$

for $s_0 \leq s \leq s_1$. We define $\xi(s) = v(s)/\phi_p(u(s)) - \Gamma_p$. Then, using (1.8), we have

$$\begin{aligned}\dot{\xi}(s) &\geq -(p-1)\,H(\xi(s),\Gamma_p) - \sum_{k=1}^{n} \frac{\mu_p}{\mathrm{Log}_k^2(u(s)^{p/(p-1)})} \\ &\geq -(p-1)\,H(\xi(s),\Gamma_p) - \sum_{k=1}^{n} \frac{\mu_p}{\mathrm{Log}_k^2(e^s)}\end{aligned}$$

for $s_0 \leq s \leq s_1$. Since $\xi_H(s)$ is a solution of (3.15) satisfying $\xi_H(s_0) = \xi(s_0)$, we have $\xi(s) \geq \xi_H(s)$ for $s_0 \leq s \leq s_1$. Hence, by (3.17), we conclude that

$$0 < \xi_H(s_1) \leq \xi(s_1) = 0,$$

which is a contradiction. The proof is now complete. □

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